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A formula relating infinitesimal Bäcklund transformations to hierarchy generating operators

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Abstract. Let $u' = B_\eta u$ and $\mathscr{L}$ be, respectively, the elementary Bäcklund transformation and hierarchy generating operator for the AKNS equations. It is shown that $(\mathrm{d}B/\mathrm{d}\eta)(B_\eta)^{-1} = \sigma_3/(\mathscr{L} - \eta)$. A similar formula relating to the general $N \times N$ matrix spectral problem is also derived.

The well known Bäcklund transformations (BTs) are powerful tools in studying nonlinear soliton equations (Miura 1976). The BT relates intimately with the existence of infinitely many conservation laws, which in turn characterises the complete integrability. Various properties of the BT as an intrinsic symmetry of nonlinear equations have been discussed in many papers (see e.g. Adler 1981, Chen 1974, Fokas and Anderson 1979, Fokas and Fuchssteiner 1981, Hirota 1979, Tu 1982, Shadwick 1980, 1981, Wadati *et al* 1975, Wahlquist and Estabrook 1973). In physics the continuous symmetry transformations may frequently be generated by some infinitesimal transformations, and then the conservation laws follow by using variational principles. Although it has been shown by Lund (Miura 1976) that in some cases (e.g. the Sine–Gordon (SG) equation) there exists a variational principle for the finite BT, i.e. it changes the Lagrangian by a total divergence, but since the BTs are discrete, such that they change an integral number of poles and (or) zeros of the scattering data or correspondingly increase or decrease an integral number of solitons, it seems of no avail to try to construct an infinitesimal BT (IBT). However, Steudel (1975) succeeded in finding the IBT for the SG and KdV equations. Noticing that more generally the positions of poles (zeros) as the parameters characterising the BT are continuous, one may construct IBT simply by manipulating these parameters. Thus we express the IBT explicitly in terms of the generating operator L of the hierarchy of nonlinear evolution equations (cf formula (A)). Here the crucial point is that, while the ordinary infinitesimal symmetry generators are solution independent operators, the generators $(L_A)^n$ for conservation laws in soliton equations depend on the solution P (cf equation (4)), moreover the finite BT operator consists of an operator Λ depending both on the initial solution P and on the final solution P'. However, for the infinitesimal BT the final solution approaches the initial ones, so we obtain expressions which only involve L. Thus we may further obtain the infinitely many operators of a conservation

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law just by expanding this expression of IBT in series (B). The relations of the IBT with the scattering data (5), with the Riccati functions and the recurrence formula (6), become more apparent, and are consistent with the commutability of BTs. Now we turn to the derivation of IBTs.

The spectral problem under consideration is (Newell 1979)

$$\psi_x = \mathrm{i}\lambda A\psi + \mathrm{i}P\psi, \tag{1}$$

where λ is a spectral parameter and

$$\psi = (\psi_1, \ldots, \psi_N)^{\mathrm{T}}, \qquad P = (P_{jk}) \qquad (P_{jk} = 0, \text{ when } j = k)$$
$$A = \operatorname{diag}(a_1, \ldots, a_N), \qquad a_j = \text{constant}, \qquad a_j \neq a_k.$$

The (generalised) BT relating to the above spectral problem was first found for $N = 2$ by Calogero and Degasperis (1976, 1977) and then extended to the general case via different approaches (see e.g. Dodd and Bullough 1977, Dodd and Morris 1980, Konopelchenko 1980, 1981, Boiti and Tu 1982). In its most general form the BT relating the potential P' to P reads

$$\sum_{j=1}^{N} \{B_j(\Lambda_A)(H_jP' - PH_j)\}_{\mathrm{F}} = 0, \tag{2}$$

where $B_j(\lambda)$ are arbitrary entire functions, $H_j = \operatorname{diag}(\delta_{j1}, \ldots, \delta_{jN})$ and G_{D} and G_{F} represent, respectively, the diagonal and off-diagonal parts of a matrix G, i.e.

$$G_{\mathrm{D}} \equiv \operatorname{diag}(G_{11}, \ldots, G_{NN}), \qquad G_{\mathrm{F}} \equiv G - G_{\mathrm{D}}.$$

The operator Λ appearing in (2) is defined by

$$\Lambda G \equiv -\mathrm{i}G_{,X} + (GP' - PG)_{\mathrm{F}} - \mathrm{i}I(GP' - PG)_{\mathrm{D}}P' - \mathrm{i}PI(GP' - PG)_{\mathrm{D}} \tag{3}$$

and $\Lambda_A G \equiv \Lambda G_A$, where $I \equiv \int_{-\infty}^{x} \mathrm{d}x'$ and the matrix G_A is determined by the equation

$$G = [A, G_A] \equiv AG_A - G_AA.$$

By the elementary BT $B_\eta^{(k)}$ we mean the BT (2) with

$$B_k(\lambda) = \lambda - \eta, \qquad B_j(\lambda) = 1 \qquad (j \neq k).$$

It is known (Konopelchenko 1979, 1981) that the general 'soliton BT', that is the BT which changes the number of solitons, can be represented as a product of elementary BTs.

Now we construct the following infinitesimal BT:

$$P' \equiv B^{(k)}_{\eta+\varepsilon/2}(B^{(k)}_{\eta-\varepsilon/2})^{-1}P$$

where ε is an infinitesimal parameter. In view of the group property of BTs the corresponding functions $B_j(\lambda)$ are

$$B_k(\lambda) = [\lambda - (\eta + \varepsilon/2)]/[\lambda - (\eta - \varepsilon/2)], \qquad B_j(\lambda) = 1 \qquad (j \neq k).$$

Accordingly, from (2) we have

$$[\Lambda_A - (\eta + \varepsilon/2)](H_kP' - PH_k) + [\Lambda_k - (\eta - \varepsilon/2)] \sum_{\substack{j=1 \\ j\neq k}}^{N} (H_jP' - PH_j) = 0$$

or

$$2(\Lambda_A - \eta)[(P' - P)/\varepsilon] - [(2H_k - 1)P' - P(2H_k - 1)] = 0.$$

Since

$$(P'-P)/\varepsilon = (1/\varepsilon)\{B^{(k)}_{\eta+\varepsilon/2}(B^{(k)}_{\eta-\varepsilon/2})^{-1}P-P\}$$

$$=[(B^{(k)}_{\eta+\varepsilon/2}-B^{(k)}_{\eta-\varepsilon/2})/\varepsilon](B^{(k)}_{\eta-\varepsilon/2})^{-1}P \xrightarrow[\varepsilon\to 0]{} (\mathrm{d}B^{(k)}_{\eta}/\mathrm{d}\eta)(B^{(k)}_{\eta})^{-1}P,$$

$$P' \xrightarrow[\varepsilon\to 0]{} B^{(k)}_{\eta}(B^{(k)}_{\eta})^{-1}P=P,$$

and $[2H_k-1, P]=2[H_k, P]$ we deduce from the above equations that

$$(\mathrm{d}B^{(k)}_{\eta}/\mathrm{d}\eta)(B^{(k)}_{\eta})^{-1}P=[1/(L_A-\eta)][H_k, P] \tag{A}$$

where

$$L\equiv\Lambda(P'=P)=-\mathrm{i}\partial_x+[\cdot, P]_{\mathrm{F}}-\mathrm{i}[I[\cdot, P]_{\mathrm{D}}, P] \tag{4}$$

is known to be the generator operator of the hierarchy of nonlinear evolution equations (NLLEs) relating to the spectral problem (1).

Formula (A) can be written in another form which relates the BT to the conserved quantities. To do this we note that (Konopelchenko 1981)

$$[1/(L_A-\eta)][H_k, P]=-\mathrm{i}[A, (\delta/\delta P^{\mathrm{T}})\mathrm{Tr}(H_k \ln S_{\mathrm{D}}(\lambda))], \tag{5}$$

where P^{T} stands for the transpose of P, $\mathrm{Tr}(G)$—the trace of G, and $\delta/\delta P$ is the variational derivative. The matrix $S(\lambda)$ is the transition matrix of the spectral problem (1), for which

$$\ln S_{\mathrm{D}}(\lambda)=\sum_{n=1}^{\infty}\eta^{-n}C^{(n)}$$

holds with

$$C^{(n)}\equiv\mathrm{diag}(C^{(n)}_{11},\ldots, C^{(n)}_{NN})=-\mathrm{i}\int_{-\infty}^{\infty}(PR^{(n)})_{\mathrm{D}}\,\mathrm{d}x$$

being the infinite number of conserved quantities which can be calculated in a recurrent way

$$R^{(1)}=-P_A,$$

$$R^{(n+1)}=\left(-\mathrm{i}R^{(n)}_{,x}+\sum_{j=1}^{n-1}R^{(j)}(PR^{(n-j)})_{\mathrm{D}}-\mathrm{i}(PR^{(n)})_{\mathrm{F}}\right)_A. \tag{6}$$

Substituting the above expansion of $\ln S_{\mathrm{D}}(\lambda)$ into (5) we obtain

$$\frac{1}{(L_\Lambda-\eta)}[H_k, P]=-\mathrm{i}\sum_{n=1}^{\infty}\left[A, \frac{\delta}{\delta P^{\mathrm{T}}}C^{(n)}_{kk}\right]\eta^{-n}$$

which together with formula (A) yields

$$\frac{\mathrm{d}B^{(k)}_{\eta}}{\mathrm{d}\eta}(B^{(k)}_{\eta})^{-1}P=-\mathrm{i}\sum_{n=1}^{\infty}\left[A, \frac{\delta}{\delta P^{\mathrm{T}}}C^{(n)}_{kk}\right]\eta^{-n}. \tag{B}$$

Let us now turn to the special case when $N=2$ and

$$A=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \qquad P=-\mathrm{i}U, \qquad U\equiv\begin{pmatrix}0 & q\\ r & 0\end{pmatrix},$$

which correspond to the well known Zakharov–Shabat and AKNS case (Zakharov and Shabat 1972, Ablowitz *et al* 1974), the problem (1) and the elementary BT $B_\eta \equiv B_\eta^{(2)} = (B_\eta^{(1)})^{-1}$ then reduce, respectively, to (Konopelchenko 1979, 1981, Sasaki 1980)

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{,x} = \begin{pmatrix}-\mathrm{i}\lambda & q\\ r & \mathrm{i}\lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

and

$$B_\eta: \qquad \begin{aligned} &r_{,x} - 2\mathrm{i}r - 2\mathrm{i}\eta r' - \tfrac{1}{2}\mathrm{i}qr'^2 = 0,\\ &q_{,x} + 2\mathrm{i}q' + 2\mathrm{i}\eta q + \tfrac{1}{2}\mathrm{i}r'q^2 = 0. \end{aligned} \tag{7}$$

It is convenient in this case to identify the off-diagonal matrix G with a vector $\hat{G}$

$$G = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix} \Leftrightarrow \begin{pmatrix}b\\ a\end{pmatrix} = \hat{G};$$

then the operator L_A will correspond to an operator $\hat{L}_A$ with $\hat{L}_A\hat{G} = \widehat{L_A G}$:

$$\begin{aligned} L_A G^{(A)} &= -\mathrm{i}G_{A,x} - \mathrm{i}(I[G_A, P], P)\\ &= \frac{1}{2\mathrm{i}}\begin{pmatrix}0 & -a_{,x} + 2qI(qb+ra)\\ b_{,x} - 2rI(qb+ra) & 0\end{pmatrix}\\ &\Leftrightarrow \frac{1}{2\mathrm{i}}\begin{pmatrix}b_{,x} - 2rI(qb+ra)\\ -a_{,x} + 2qI(qb+ra)\end{pmatrix} = \frac{1}{2\mathrm{i}}\begin{pmatrix}D - 2rIq & -2rIr\\ 2qIq & -D + 2qIr\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} \equiv \hat{L}_A\hat{G}, \end{aligned}$$

or

$$\mathscr{L} \equiv \sigma_3\hat{L}_A\sigma_3 = \frac{1}{2\mathrm{i}}\begin{pmatrix}D - 2rIq & 2rIr\\ -2qIq & -D + 2qIr\end{pmatrix}. \tag{8}$$

Substituting $P = -\mathrm{i}U$ in the formula (A) with $N = k = 2$ we obtain

$$(\mathrm{d}B_\eta/\mathrm{d}\eta)B_\eta^{-1}U = [1/(L_A - \eta)][H_2, U] = -[1/(L_A - \eta)]\sigma_3 U.$$

Applying on both sides the operation $\hat{}$, and taking into account the fact that $\widehat{\sigma_3 U} = -\sigma_3\hat{U}$, $\hat{U} = (r, q)^{\mathrm{T}} \equiv u$, we find

$$(\mathrm{d}B_\eta/\mathrm{d}\eta)B_\eta^{-1}u = [1/(\hat{L}_A - \eta)]\sigma_3 u,$$

where the BT $u' = B_\eta u$ is defined again by (7). The above equation can also be written as

$$(\mathrm{d}B_\eta/\mathrm{d}\eta)B_\eta^{-1}u = \sigma_3[1/(\mathscr{L} - \eta)]u \tag{A$'$}$$

with $\mathscr{L} = \sigma_3\hat{L}_A\sigma_3$ being defined by (8).

In the same manner, by substituting $P = -\mathrm{i}U$ into (B) and setting $I_n \equiv C_{22}^{(N)}$ we obtain

$$\frac{\mathrm{d}B_\eta}{\mathrm{d}\eta}B_\eta^{-1}U = \mathrm{i}\sum_{n=1}^{\infty}\left[A, \frac{\delta}{\delta U^{\mathrm{T}}}I_n\right]\eta^{-n} = -2\mathrm{i}\sigma_3\sum_{n=1}^{\infty}\left(\frac{\delta}{\delta U^{\mathrm{T}}}I_m\right)\eta^{-n},$$

or, after applying the operator Λ,

$$\frac{\mathrm{d}B_n}{\mathrm{d}\eta}B_\eta^{-1}u = 2\mathrm{i}\sigma_3\sum_{n=1}^{\infty}\begin{pmatrix}(\delta/\delta q)I_n\\ (\delta/\delta r)I_n\end{pmatrix}\eta^{-n} = -2\sigma_2\sum_{n=1}^{\infty}\begin{pmatrix}(\delta/\delta r)I_n\\ (\delta/\delta q)I_n\end{pmatrix}\eta^{-n}.$$

Denoting by J the symplectic matrix $-2\sigma_2$ and $\delta/\delta u = (\delta/\delta r, \delta/\delta q)^{\mathrm{T}}$, we obtain

$$(\mathrm{d}B_\eta/\mathrm{d}\eta)B_\eta^{-1}u = \sum_{n=1}^{\infty} J\left(\frac{\delta}{\delta u}I_n\right)\eta^{-n} \tag{B'}$$

which is, of course, compatible with (A'), since $\sigma_3\mathscr{L}^{n-1}u = -J(\delta/\delta u)I_n$. Note also that, from (6), the conserved quantities $\{I_n\}$ can be determined by the recursion relation

$$I_n = -\int_{-\infty}^{\infty} rZ_n \,\mathrm{d}X, \qquad Z_1 = -\tfrac{1}{2}\mathrm{i}q, \qquad Z_{n+1} = \left(Z_{n,X} + \sum_{j=1}^{n-1} Z_j Z_{n-j}\right),$$

which are just the series expansion of the Riccati function $\frac{1}{2}\mathrm{i}q'$ (cf (7) and Sasaki 1980).

For simplicity we have restricted ourselves to the elementary BT with one pole only. It is straightforward to generalise to an arbitrary general BT with both poles and zeros. It is easy to apply to the reduced ($r = q$ or $r = 1$) cases also.

In the case of dual symmetric systems such as the nonlinear σ model (Pohlmeyer 1976), self-dual Yang–Mills field (Morris 1980, Prasad *et al* 1979) and Ernst equations we have found that the IBT are space–time dependent 'isospin' rotations, whose rotation axis satisfies the matrix Riccati equation (Hou 1981).

As pointed out by one of the referees, the entries in operator P should be distinct to avoid the collapse of the Bäcklund transformation (Kaup 1976).

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